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THE INVERSE PROBLEM OF ACOUSTIC-WAVE SCATTERING FOR THIN ACOUSTICALLY RIGID BODIES*

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A description of an analytical algorithm for determining the shape of the scatterer with acoustically rigid walls is given for low-frequency scattering of plane acoustic waves. It is assumed that the amplitude of the scattering is known in a direction close to the direction of backward scattering, specified on a discrete set of probing wave numbers. The incident wavelength is of the order of the length of the scatterer and is much greater than its thickness.

A problem similar to this was discussed in /1/, where, however, it was assumed that the amplitude of the scattered plane waves is known on the surface of the unit sphere, and in a continuous spectrum of fairly small wave numbers. It was also assumed that there is a priori information available on the minimum radius of the sphere containing the scatterer inside it, and that the constant which limits the potential gradient of the velocity of the liquid outside the sphere is known for the problem of determining the streamlines of a vortex-free liquid.

Suppose that an absolutely rigid body D_* with a boundary S, described by the equation

$$r = \varepsilon F(t, \varphi),\ 0 \leqslant t \leqslant a,\ \ 0 \leqslant \varphi \leqslant 2\pi,\ F(0, \varphi) = F(a, \varphi) = 0 \tag{1}$$

is situated in a space filled with an acoustic medium, where r, φ, t are cylindrical coordinates with origin at the point O, and $\varepsilon > 0$ is a small parameter. The function $F^2(t, \varphi)$ is assumed to be integrable with a square on the surface of the unit cylinder $G = \{0 \leqslant t \leqslant a, 0 \leqslant \varphi \leqslant 2\pi, r = 1\}$. We will assume that a plane wave $u_n(\mathbf{x}) = A_0 \exp[ik(\mathbf{l}, \mathbf{x})]$ is incident from infinity on the body D_* (here and henceforth the time factor $\exp[-ika\tau]$ is omitted), where $\mathbf{l} = (l_1, l_2, l_3)$ is the unit vector indicating the direction of propagation of the wave, A_0 is its amplitude, $\mathbf{x} = (x_1, x_2, x_3)$ is the radius vector of an arbitrary point of space drawn from the point O, (,) is scalar multiplication, and k is the wave number, which is assumed to real and positive. The scattered field $u_p(\mathbf{x})$ satisfies the Helmholtz equation in the exterior D of the body D_*, and the boundary condition

$$(\Delta + k^2)\, u_p(\mathbf{x}) = 0;\ \ du_p/dn = -du_n/dn,\ \ \mathbf{x} \in S \tag{2}$$

and also the Sommerfeld radiation condition, which can be written in the form

$$u_p(\mathbf{x}) = -4\pi\, |\mathbf{x}|^{-1} \exp(ik\,|\mathbf{x}|)\, f(k; \mathbf{l}, \mathbf{v}) + o(|\mathbf{x}|^{-1})$$
$$(|\mathbf{x}| \to \infty)$$

Here Δ is the Laplace operator, d/dn is the derivative with respect to the direction of the external normal to S, $|\mathbf{x}| = (\mathbf{x}, \mathbf{x})^{1/2}$ is the length of the vector $\mathbf{x}$, $\mathbf{v} = \mathbf{x}\,|\mathbf{x}|^{-1}$ is the unit vector in the direction x, and $f(k; \mathbf{l}, \mathbf{v})$ is the scattering amplitude.

Consider the problem of determining the function $\varepsilon F(t, \varphi)$ from the scattering amplitude known in one of the directions in space, specified by a discrete set of wave numbers.

The solution of the direct problem (1), (2) is unique and can be represented for $x \in S$ as the solution of the integral equation /2/

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$$u_p(\mathbf{x}) = \frac{1}{4\pi}\int_S [u(\mathbf{y}) - u(\mathbf{x})]\frac{\partial}{\partial n_y}\frac{1}{|\mathbf{x}-\mathbf{y}|}dS_y + \frac{1}{4\pi}\int_S u(\mathbf{y})\frac{\partial}{\partial n_y}\frac{e^{ik|\mathbf{x}-\mathbf{y}|}-1}{|\mathbf{x}-\mathbf{y}|}dS_y \qquad (3)$$

$$(u(\mathbf{x}) = u_n(\mathbf{x}) + u_p(\mathbf{x}),\ \ d/dn_y \equiv d/dn(\mathbf{y}))$$

where $dS_y = dS(\mathbf{y})$ is an element of the area of the surface S at the point $\mathbf{y} = (y_1, y_2, y_3)$. We obtain (3)

$$f(k; \mathbf{l}, \mathbf{v}) = -\int_S u(\mathbf{y})\exp[-ik(\mathbf{v}, \mathbf{y})]\,dS_y$$

If we have the inequality

$$\frac{k^2}{2\pi}\int_S dS_y \leqslant M < 1$$

where M is a certain constant /2/, the solution of (3) can be represented in the form of a Neumann series. In particular, the zeroth approximation is given by the equation $u(\mathbf{x}) = u_n(\mathbf{x})$ on S, while the first term of the Neumann series for the scattering amplitude, using (1), can be written in the form

$$f_0(k; \mathbf{l}, \mathbf{v}) = A_0k^2[1-(\mathbf{v}, \mathbf{l})] \times \int_G d\varphi\, dt \int_0^{\varepsilon F(t,\varphi)} d\rho\,\rho\exp\{ik\rho[(l_1-\nu_1)\cos\varphi + (l_2-\nu_2)\sin\varphi] + ikt(l_3-\nu_3)\} \qquad (4)$$

Of course the function $f_0(k; \mathbf{l}, \mathbf{v})$ contains the principal term of the asymptotic form of the scattering amplitude, and corresponds to taking into account the singly reflected waves assuming that $ka\varepsilon$ is small. Then $f_0(k; \mathbf{l}, \mathbf{l}) = 0$, i.e., to a first approximation there is no forward scattering. As $k\mathbf{l} \to 0$ we have

$$f_0(k; \mathbf{l}, \mathbf{v}) \simeq f(k; \mathbf{l}, \mathbf{v}) = A_0k^2[1-(\mathbf{v}, \mathbf{l})]V_0 + o(k^2) \qquad (5)$$

Here V_0 is a constant which is independent of the scattering direction and is equal to the volume of the scatterer.

Then, if the scattering amplitude is known for a discrete set of fairly small probing wave numbers, relation (5) enables one to calculate the volume of the scatterer.

By considering the integral over ρ in (4) as a function of $\varepsilon F(t, \varphi)$ and representing it by a Taylor series in the neighbourhood of zero, we obtain

$$f_0(k; \mathbf{l}, \mathbf{v}) = A_0(k\varepsilon)^2 2^{-1}[1-(\mathbf{v}, \mathbf{l})] \times \int_G F^2(t, \varphi)\exp[ikt(l_3-\nu_3)]\,dt\,d\varphi + o(\varepsilon^2) \qquad (6)$$

Note that expansion (6), which is uniform with respect to ε, will also be finite if $l_i = \nu_i$, $i = 1, 2$.

Suppose the vectors $\mathbf{l}$ and $\mathbf{v}$ are fixed so that $\nu_3 \neq 0, 1$, and the vector $\mathbf{v}$ is close to the backward-scattering direction, i.e., $\mathbf{v} = -\mathbf{l}$, while the function $f_0(k; \mathbf{l}, \mathbf{v})$ is known when $k = k_m$ $(m = 1, 2, \ldots, N)$. Then, from (6) we obtain the following set of integral equations of the first kind in terms of the function $\varepsilon^2F^2\cdot(t, \varphi)$:

$$\varepsilon^2\int_G F^2(t, \varphi)\exp[i\alpha_m t]\,dt\,d\varphi = f_m, \quad m = 1, 2, \ldots, N \qquad (7)$$

$$(f_m = 2f_0(k_m; \mathbf{l}, \mathbf{v})\{A_0k_m^2[1-(\mathbf{v}, \mathbf{l})]\}^{-1},\ \alpha_m = k_m(l_3-\nu_3))$$

From the mathematical point of view the solution of (7) is an ill-posed problem /3/, which, generally speaking, has an innumerable set of solutions, and these solutions are unstable. The problem arises of regularizing it and making it solvable, for example, by introducing a priori information on the observed object.

In fact, when $F(t, \varphi) = F(\varphi)$ we obtain the problem of determining the form of the region from its volume, reduced to a certain value, the solution of which is not unique. Hence we will put $F(t, \varphi) = \lambda_1(t)\lambda_2(\varphi)$, and the function $\lambda_2(\varphi)$ will be assumed known. We then obtain from (10)

$$\int_0^a g(t)\exp[i\alpha_m t]\,dt = \omega_m, \quad m = 1, 2, \ldots, N \qquad (8)$$

$$g(t) = \varepsilon^2\lambda_1^2(t), \quad \omega_m = f_m\Big(\int_0^{2\pi}\lambda_2^2(\varphi)\,d\varphi\Big)^{-1}$$

It can be shown that if the numbers k_m $(m = 1, 2, \ldots, N;\ N \to \infty)$ form a set having a point where bunching occurs at zero, then from data on the scattering ω_m the function $g(t)$ from the set of integral equations (8) can be found uniquely.

The normal pseudo-solution /3/ or (8) for any finite N can be represented in the form /4/

$$g(t) = \sum_{m=1}^{N} A_m \exp[-i\alpha_m t] \tag{9}$$

and in a quadratic metric approaches the accurate solution as $N \to \infty$. The coefficients A_m $(m = 1, 2, \ldots, N)$ are a unique and stable solution of the linear set of algebraic equations

$$\sum_{n=1}^{N} A_n \frac{\exp[ia(\alpha_m - \alpha_n)] - 1}{i(\alpha_m - \alpha_n)} = \omega_m, \quad m = 1, 2, \ldots, N \tag{10}$$

However, as $N \to \infty$, system (10) becomes degenerate and it is necessary to use regularization methods /3/.

As an example, consider the problem of determining the form of a thin prolate spheroid. Suppose

$$\varepsilon^2 F^2(t, \varphi) = g(t) = \varepsilon^2 t(a - t), \quad a = d(1 - \varepsilon^2)^{-1/2}$$

(Fig.1), the continuous curve), where d is the distance between the foci.

Then, we obtain from (6)

$$f_0(k; \mathbf{l}, \mathbf{v}) = A_0\varepsilon^2[1 - (\mathbf{v}, \mathbf{l})]\pi \exp(i\alpha)\frac{2a}{(l_3 - v_3)^2}\left(-\cos\alpha + \frac{\sin\alpha}{\alpha}\right) \tag{11}$$
$$\alpha = {}^1/_2\, ka(l_3 - v_3)$$

In Fig.2 we show graphs of $|f(k; \mathbf{l}, \mathbf{v})|(2\pi d)^{-1}$ as a function of $\mathbf{v}$ in the Ox_2x_3 plane for $kd = 2$ and $\mathbf{l} = (0, 0, 1)$, taken from /5/.

Curves *1* and *2* correspond to $\varepsilon = 0, 1$ and $\varepsilon = 0, 2$, where the dots denote the results of calculations using (11).

Similar curves are shown in Fig.3 for $kd = 6$, $\mathbf{l} = (0, 1, 0)$.

As can be seen from Figs.2 and 3, Eq.(11) gives a satisfactory approximation to the exact value $f(k; \mathbf{l}, \mathbf{v})$ in the half-plane of backward scattering for small $ka\varepsilon$.

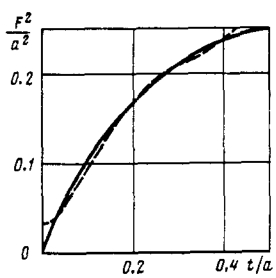

Fig.1

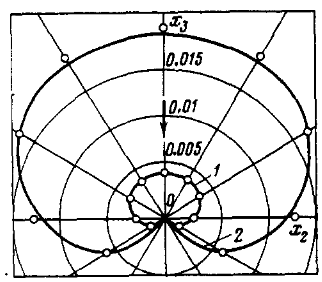

Fig.2

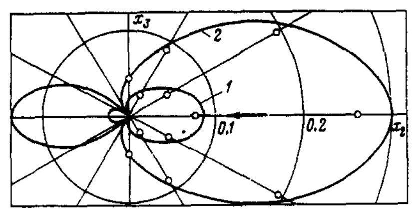

Fig.3

Suppose the incident wave propagates along the axis of rotation of the spheroid - the Ox_3 axis: $u_n(\mathbf{x}) = A_0 \exp(ikx_3)$. We will also assume that $\mathbf{v} = -\mathbf{l}$. Then from (8) and (11) we obtain /6/

$$\omega_1 = V_0/\pi = \varepsilon^2 a^3 6^{-1}, \quad \omega_2 = -\varepsilon^2 a^3 (2\pi^2)^{-1}$$
$$\omega_3 = -\varepsilon^2 a^3 (8\pi^2)^{-1}, \quad \omega_4 = -\varepsilon^2 a^3 (18\pi^2)^{-1}$$

which corresponds to $k_1 a \to 0$, $k_2 a = \pi$, $k_3 a = 2\pi$, $k_4 a = 3\pi$.

Obviously in this case the scattering amplitude does not change if the vectors and $\mathbf{v}$ are interchanged. We then obtain from (9) and (10) (Fig.1, the dashed curve)

$$\varepsilon^{-2} g(t) = F^2(t) = a^2\left[\frac{1}{6} - \frac{1}{\pi^2}\cos\left(2\pi\frac{t}{a}\right) - \frac{1}{4\pi^2}\cos\left(4\pi\frac{t}{a}\right) - \frac{1}{9\pi^2}\cos\left(6\pi\frac{t}{a}\right)\right]$$

In view of the fairly good agreement between the curves in Fig.1 we can assert that to determine the form of a thin acoustically rigid prolate spheroid three values of ω_m $(m = 2, 3, 4)$ are sufficient to determine its volume.

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A CLASS OF EXACT SOLUTIONS WITH A UNIFORM DEFORMATION IN GAS DYNAMICS*

S.A. POSLAVSKII and I.S. SHIKIN

A new exact solution is obtained describing the motion of a rotating gas ellipsoid in which the ratio of the semiaxes remains constant (with an adiabatic index of 5/3). The structure of this solution is in a certain sense similar to the structure of the solutions for an ellipsoid of a uniform ideal incompressible selfgravitating liquid, obtained in papers on the theory of equilibrium figures (see 1, 2/).

The adiabatic motions of an ideal gas with a non-uniform deformation were first studied by Sedov /3, 4/, who obtained an exact non-stationary solution in the uniform case. Ovsyannikov /5/ showed that in a more general formulation the problem reduces to a set of nine second-order ordinary differential equations. This system allows seven first integrals connected with the conservation of energy and momentum of the gas cloud and "freezing in" of the vortex /5, 6/. An eighth integral was obtained in /7/ with an adiabatic index of 5/3, and an exact solution of the problem of the dispersion of a non-rotating gas ellipsoid of rotation in a vacuum was found. The problem of the motion of a rotating spheroid was considered in /8/. A qualitative investigation in the general case of motion with a uniform deformation of a triaxial gaseous ellipsoid was carried out in /9/.**

1. Solutions with non-uniform deformation are characterized by a linear depedence of the Euler coordinates on the Lagrangians

$$x_\alpha = M_{\alpha\beta}(t)\,\xi_\beta \tag{1.1}$$

Here and hence for the Greek subscripts take values from 1 to 3, and summation is carried out over the repeated indices.

For an ellipsoidal distribution of the density and pressure, the adiabatic motions of a gas are described by the equations

$$\ddot{M}_{\alpha\beta} = -\frac{\varepsilon}{D^{\gamma-1}}(M^{-1})_{\beta\alpha};\quad D = \det M,\quad \gamma = \frac{c_p}{c_v},\quad \varepsilon = \mathrm{const} \tag{1.2}$$

The density and pressure are given by the equations

$$\rho = \frac{\rho_0(\sigma)}{D},\quad p = \frac{p_0(\sigma)}{D^\gamma};\quad p_0(\sigma) = p_0(0) + \int_0^\sigma \rho_0(\lambda)\,d\lambda,\qquad \sigma = \frac{\varepsilon}{2}(\xi_1^2 + \xi_2^2 + \xi_3^2) \tag{1.3}$$

where $\rho_0(\sigma)$ is an arbitrary function. For a finite ellipsoid with boundaries $\xi_1^2 + \xi_2^2 + \xi_3^2 = 1$ the function $\rho_0(\sigma)$ vanishes outside it.

If $\varepsilon < 0$, the pressure falls with distance from the centre of the ellipsoid; if $\varepsilon > 0$, it increases. The first case may correspond, for example, to the motion of a gas cloud in a vacuum, and the second may correspond to the motion of an ellipsoid acted upon by an external pressure.

The matrix M can be represented in the form

$$M = Q_1 A Q_2 \tag{1.4}$$

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**A single parametric family of exact solutions for a gas spheroid with a constant ratio of the semiaxes was obtained by Bogoyavlenskii in a paper entitled "The oscillatory expansion of a gas cloud in a vacuum", Preprint of the Institute of Theoretical Physics Academy of Sciences of the USSR, Chernogolovka, 1975.